Math 10/11 Honors Challenging Sequences and Series Questions:

For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

AMC 10

The increasing sequence 2, 3, 5, 6, 7, 10, 11 …….. consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term.

Define the sequence A_0, A_1, A_2, and so on by $A_0 = A_1 = 1$, and $A_n = 2A_{n-1} + A_{n-2}$ for $n \geq 2$. Let $x = 1/3$. Calculate

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n + \cdots .$$

Manipulate "infinite sums" freely, assume they behave algebraically like finite sums.

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The sequence a_0, a_1, a_2, and so on is defined by $a_0 = 2$ and $a_{n+1} = (2a_n + 1)/(a_n + 2)$ for $n \geq 0$. Find an explicit formula for a_n, and prove that the formula is correct.

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The increasing sequence: 1, 3, 4, 9, 10, 12, 13, ….. consists of all those positve integers which are powers of 3 or sums of distince powers of 3. Find the 100th term of this sequence

AIME 198X

. Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)^3},$$

given Euler's beautiful result that

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

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For the sake of notation let $T(n) = n + S(n) + S(S(n))$. Obviously $n < 2007$. Then the maximum value of $S(n) + S(S(n))$ is when $n = 1999$, and the sum becomes $28 + 10 = 38$. So the minimum bound is 1969. We do casework upon the tens digit:

Case 1: $196u \implies u = 9$. Easy to directly disprove.

Case 2: $197u$. $S(n) = 1 + 9 + 7 + u = 17 + u$, and $S(S(n)) = 8 + u$ if $u \leq 2$ and $S(S(n)) = 2 + (u - 3) = u - 1$ otherwise.

Subcase a: $T(n) = 1970 + u + 17 + u + 8 + u = 1995 + 3u = 2007 \implies u = 4$. This exceeds our bounds, so no solution here.
Subcase b: $T(n) = 1970 + u + 17 + u + u - 1 = 1986 + 3u = 2007 \implies u = 7$. First solution.

Case 3: $198u$. $S(n) = 18 + u$, and $S(S(n)) = 9 + u$ if $u \leq 1$ and $2 + (u - 2) = u$ otherwise.

Subcase a: $T(n) = 1980 + u + 18 + u + 9 + u = 2007 + 3u = 2007 \implies u = 0$. Second solution.
Subcase b: $T(n) = 1980 + u + 18 + u + u = 1998 + 3u = 2007 \implies u = 3$. Third solution.

Case 4: $199u$. But $S(n) > 19$, and $n + S(n)$ clearly sum to > 2007.

Case 5: $200u$. So $S(n) = 2 + u$ and $S(S(n)) = 2 + u$ (recall that $n < 2007$), and $2000 + u + 2 + u + 2 + u = 2004 + 3u = 2007 \implies u = 1$. Fourth solution.

In total we have $4(\mathrm{D})$ solutions, which are 1977, 1980, 1983, and 2001.

Because there aren't that many perfect squares or cubes, let's look for the smallest perfect square greater than 500. This happens to be $23^2 = 529$. Notice that there are 23 squares and 8 cubes less than or equal to 529, but 1 and 2^6 are both squares and cubes. Thus, there are $529 - 23 - 8 + 2 = 500$ numbers in our sequence less than 529. Magically, we want the $500th$ term, so our answer is the smallest non-square and non-cube less than 529, which is $\boxed{528}$.

Solution. The calculation imitates the standard way to sum the infinite geometric series $1 + x + x^2 + \cdots$. Recall that if we let that sum be $G(x)$, then $G(x) - xG(x) = 1$ (almost everything cancels). It follows that $G(x) = 1/(1-x)$. So let our sum be $S(x)$. Multiply $S(x)$ by $2x$, and subtract from the expression for $S(x)$. Gathering like powers of x together, we obtain

$$S(x) - 2xS(x) = A_0 + (A_1 - 2A_0)x + (A_2 - 2A_1)x^2 + (A_3 - 2A_2)x^3 + (A_4 - 2A_3)x^4 + \cdots.$$

If $n \geq 2$, then $A_n - 2A_{n-1} = A_{n-2}$. Using this, we find that

$$S(x) - 2xS(x) = 1 - x + A_0x^2 + A_1x^3 + A_2x^4 + \cdots = 1 - x + x^2S(x).$$

A little manipulation now gives $S(x) = (1-x)/(1-2x-x^2)$. When $x = 1/3$, this is equal to 3.

Comment. Things are somewhat more messy looking if from the beginning we work with $1/3$ rather than x. This increases the probability of error, and more importantly makes it more likely that a nice structural pattern will be missed. Quite often in problems, even when specific numbers are mentioned, it can be useful to replace them by letters. Any "algebra" will look much neater, and one may get a general result. Working with specific numbers from the beginning may be necessary, but it should be postponed if possible. In particular, premature use of the calculator can hide vital structural information.

As instructed, we operated "formally" on the series, ignoring issues of convergence. It turns out that our series converges if $|x| < \sqrt{2} - 1$, which (no accident!) is one the roots of the equation $1 - 2x - x^2 = 0$. That root is roughly 0.4142, and $1/3$ is safely smaller. A proof that there is convergence at $x = 1/3$ is not hard. It is enough to show (say by induction) that $A_n < 0.35^n$ if n is large enough.

Solution. It is useful to experiment. We can without much trouble calculate the first few a_i. We find that $a_1 = 5/4$, $a_2 = 14/13$, $a_3 = 41/40$. We can see the beginnings of a possible pattern: the numerator (at least so far) is 1 more than the denominator. The number a_0 also fits the pattern, after we note that $a_0 = 2/1$.

Here is a somewhat less obvious element of the pattern. Note that the sum of the numerator and denominator, for the first few terms, is 3, 9, 27, and 81. These are the powers of 3. So we may want to guess that the numerator of a_n is $(3^{n+1}+1)/2$ and the denominator is $(3^{n+1}-1)/2$. If we divide, and for simplicity cancel the 2s, we have the conjecture that

$$a_n = \frac{3^{n+1}+1}{3^{n+1}-1}.$$

We can prove the result by Mathematical Induction, a very important idea. But we will (sort of) sidestep doing a formal induction. Let $b_n = (3^{n+1}+1)/(3^{n+1}-1)$. We would like to show that $a_n = b_n$ for all n.

An easy calculation shows that $b_0 = 2$. We will show that

$$b_{n+1} = \frac{2b_n + 1}{b_n + 2}.$$

Like most identities, this is quite easy to prove. Note that

$$2b_n + 1 = 2\frac{3^{n+1}+1}{3^{n+1}-1} + 1.$$

Rewrite all of the terms in base 3. Since the numbers are sums of *distinct* powers of 3, in base 3 each number is a sequence of 1s and 0s (if there is a 2, then it is no longer the sum of distinct powers of 3). Therefore, we can recast this into base 2 (binary) in order to determine the 100th number. 100 is equal to $64+32+4$, so in binary form we get 1100100. However, we must change it back to base 10 for the answer, which is $3^6+3^5+3^2=729+243+9=\boxed{981}$.